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Study of three-dimensional unsteady Oseen flow

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In the present paper, it is intended to give the elementary solutions of three-dimensional unsteady Oseen flow when unsteady concentrated lift and/or drag is applied in the flow field. It is shown that the pressure fields due to concentrated impulsive lift and/or drag can be represented by an impulsive pressure doublet in the direction of the applied force and the corresponding velocity fields by diffusing free doublets in the direction of the external force that are shed from the location of the force application and convected downstream with otherwise uniform velocity. It is also confirmed that combination of the elementary solutions given in the present paper yields the two-dimensional ones.

1. Introduction

The elementary solutions of two-dimensional unsteady flows based on the Oseen approximation are given for the cases where sinusoidally varying concentrated lift (Shen & Crimi 1965), impulsive lift and/or drag, and sinusoidally varying lift and/or drag (Murata, Miyake & Tsujimoto 1977) are applied at a point in the flow field. The unsteady Oseen flows around an aerofoil and through a cascade are obtained by superposing these elementary solutions (Tsujimoto *et al.* 1978).

In order to describe three-dimensional Oseen flows around three-dimensional bodies, the elementary solution for three-dimensional flow is needed. The present report gives the exact solutions of that sort for the cases where impulsive and/or sinusoidally varying concentrated forces are applied perpendicular or parallel to an otherwise uniform stream.

Any fluctuation of the force in time can be decomposed into a linear superposition of delta functions and hence the elementary solution for the case of an impulsive concentrated force is the basis of the unsteady problem. The solution for the case of a sinusoidally varying concentrated force at a point can be derived as an extension of that for an impulsive force. Moreover, an interesting feature of unsteady Oseen flow with a sinusoidally varying force manifests itself in its structure, i.e. the flow can be further decomposed into elementary parts which have a clear physical meaning.

2. Fundamental equations of three-dimensional Oseen flow and of the pressure field

Consider the flow when a small concentrated force $(X, Y, 0)$ is applied at the origin of the co-ordinate system in an otherwise uniform stream in the x direction. The fundamental equations for an incompressible fluid are

$$L_c(v_x) = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad L_c(v_y) = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad L_c(v_z) = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{1)–(3}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{4}$$

where L_c is an operator defined as

$$L_c = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right),$$

v_x, v_y and v_z are the velocity components of the perturbed flow, p is the static pressure and ρ the density of the fluid.

The Oseen flow around an oscillating body immersed in a uniform stream can be described using the elementary solutions given below. The total velocity field is obtained as the superposition of the steady Oseen flow $(U+u, v, w)$ and the unsteady perturbed flow (v_x, v_y, v_z). The convection terms included in (1)–(3) are $U(\partial v_x/\partial x)$ etc., which means that the terms $u(\partial v_x/\partial x)$, etc. are omitted. Since $u = -U$ holds on the wall, this omission leads to over-estimation of the convection terms, and in consequence, the error becomes severe as the Reynolds number increases. However, the error due to this approximation may be smaller in unsteady flow than in steady flow, because the acceleration term is the dominant inertia term in the unsteady momentum equation.

Differentiating (1), (2) and (3) with respect to x, y and z, respectively, and adding them, one obtains

$$\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}, \tag{5}$$

which means that the viscosity has no influence on the pressure field.

Now consider the case in which a concentrated impulsive lift force is applied perpendicular to the flow at the origin of the co-ordinate system, i.e.

$$X = 0, \quad Y = Y_0\,\delta(x)\,\delta(y)\,\delta(z)\,\delta(t), \tag{6}$$

where $\delta(x)$ is Dirac's impulse function, defined by $\delta(x) = 0$ when $x = 0$ and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Substitution of (6) into (5) yields

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\partial}{\partial y}\{\rho Y_0(x)\,\delta(x)\,\delta(y)\,\delta(z)\,\delta(t)\},$$

the solution of which is

$$p = \frac{\rho Y_0}{4\pi}\frac{y}{R^3}\,\delta(t), \tag{7}$$

where $R^2 = x^2+y^2+z^2$. When the concentrated lift varies sinusoidally in time as

$$X = 0, \quad Y = Y_0\delta(x)\,\delta(y)\,\delta(z)\exp(i\omega t)$$

then the pressure field is given by (7) with $\delta(t)$ replaced by $\exp(i\omega t)$, where i is the imaginary unit.

When a concentrated impulsive drag is applied at the origin, i.e.

$$X = X_0\delta(x)\,\delta(y)\,\delta(z)\,\delta(t), \quad Y = 0,$$

then the pressure field becomes

$$p = \frac{\rho X_0}{4\pi}\frac{x}{R^3}\delta(t). \tag{8}$$

When the concentrated drag varies sinusoidally, the pressure field is given by (8) with $\delta(t)$ replaced by $\exp(i\omega t)$. The case of drag produced at the origin is more conveniently treated in a cylindrical co-ordinate system, for which the fundamental equations become

$$L_p(v_x) = -\frac{1}{\rho}\frac{\partial p}{\partial x}+X, \tag{9}$$

$$L_p(v_r)+\nu\frac{v_r}{r^2} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{10}$$

$$\frac{\partial v_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0, \tag{11}$$

where

$$L_p = \frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\nu\left\{\frac{\partial^2}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right\}.$$

The velocity fields corresponding to the above-mentioned pressure field will be discussed in the following sections.

3. Flow induced by impulsive concentrated lift

Now consider a function f defined as

$$f = \frac{1}{R_1}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}, \tag{12}$$

where $R_1^2 = (x-Ut)^2+y^2+z^2 = (x-Ut)^2+r^2$ and erfc is the complementary error function, defined as

$$\operatorname{erfc}\eta = \frac{2}{\pi^{\frac{1}{2}}}\int_\eta^\infty \exp(-x^2)\,dx = 1-\frac{2}{\pi^{\frac{1}{2}}}\int_0^\eta \exp(-x^2)\,dx = 1-\operatorname{erf}\eta.$$

Here $\operatorname{erf}\eta$ is the error function and the following relations hold:

$$\operatorname{erfc}0 = 1, \quad \operatorname{erfc}\infty = 0.$$

It is easily verified that

$$L_c(f) = 0, \quad L_p(f) = 0. \tag{13}$$

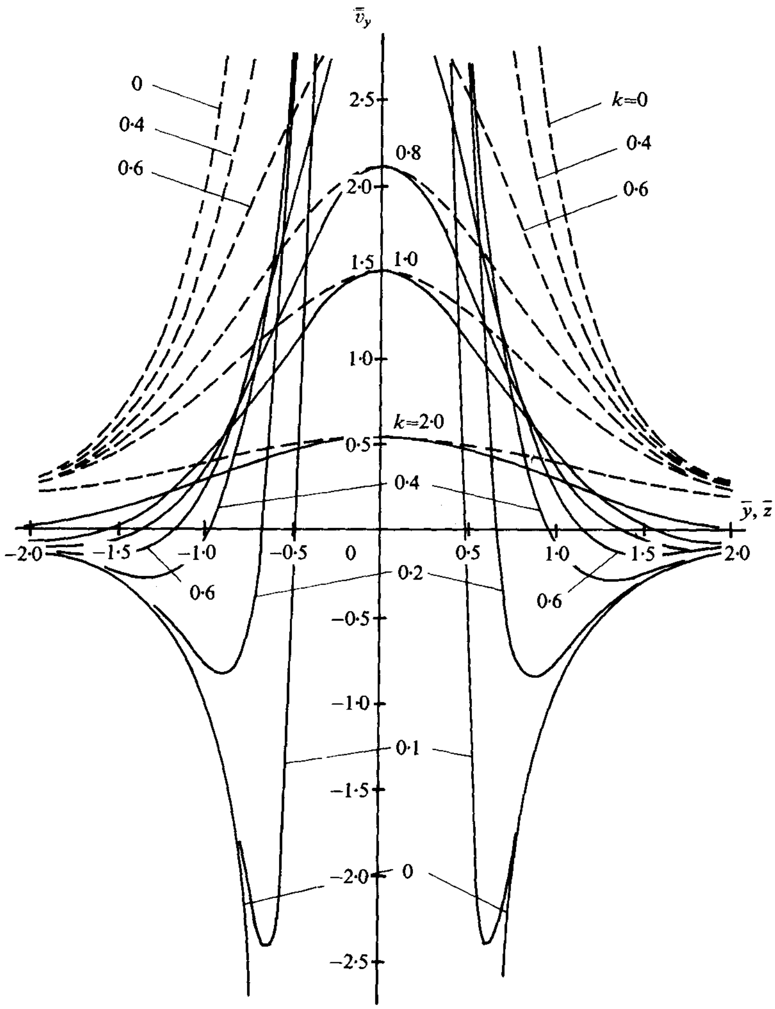

FIGURE 1. Velocity distributions induced by concentrated impulsive force. ——, $\bar{v}_y$ *vs.* $\bar{z}$; – – –, $\bar{v}_y$ *vs.* $\bar{y}$.

If the velocity (v'_x, v'_y, v'_z) is defined in terms of f as

$$v'_x = A\frac{\partial^2 f}{\partial x\,\partial y}, \quad v'_y = A\left(\frac{\partial^2 f}{\partial y^2} - \frac{1}{\nu}\frac{\partial f}{\partial t} - \frac{U}{\nu}\frac{\partial f}{\partial x}\right), \quad v'_z = A\frac{\partial^2 f}{\partial y\,\partial z}, \tag{14}$$

then in view of (13) it follows that

$$L_c(v'_x) = 0, \quad L_c(v'_y) = 0, \quad L_c(v'_z) = 0.$$

The equation of continuity is also satisfied, i.e

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} + \frac{\partial v'_z}{\partial z} = -\frac{A}{\nu}\frac{\partial}{\partial y}\{L_c(f)\} = 0.$$

The full description of v'_x, v'_y and v'_z is obtained by substituting (12) into (14):

$$v'_x = A\left[\frac{3y(x-Ut)}{R_1^5}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}-\frac{1}{(\pi\nu t)^{\frac{1}{2}}}\frac{y(x-Ut)}{R_1^4}\exp\left(-\frac{R_1^2}{4\nu t}\right)\left(3+\frac{R_1^2}{2\nu t}\right)\right], \quad (15)$$

$$v'_y = A\left[\left(-\frac{1}{R_1^3}+\frac{3y^2}{R_1^5}\right)\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{1}{(\pi\nu t)^{\frac{1}{2}}}\right.$$
$$\left.\times\exp\left(-\frac{R_1^2}{4\nu t}\right)\left\{\frac{1}{R_1^2}-\frac{3y^2}{R_1^4}+\frac{1}{2\nu t}\left(1-\frac{y^2}{R_1^2}\right)\right\}\right], \quad (16)$$

$$v'_z = A\left[\frac{3yz}{R_1^5}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}-\frac{1}{(\nu\pi t)^{\frac{1}{2}}}\frac{yz}{R_1^4}\exp\left(-\frac{R_1^2}{4\nu t}\right)\left(3+\frac{R_1^2}{2\nu t}\right)\right]. \quad (17)$$

In the limit $\nu\to 0$, the above velocities reduce to

$$v'_x = A\frac{3y(x-Ut)}{R_1^5}, \quad v'_y = A\left(-\frac{1}{R_1^3}+\frac{3y^2}{R_1^5}\right), \quad v'_z = A\frac{3yz}{R_1^5},$$

for which there exists the velocity potential

$$\phi = -Ay/R_1^3.$$

ϕ is in turn derived from another potential Φ, i.e.

$$\phi = \partial\Phi/\partial y, \quad \Phi = A/R_1.$$

Φ is the velocity potential of a point source convected by a uniform stream of velocity U, so that ϕ is a y-wise free doublet shed from the origin and convected downstream with the velocity U, ϕ being derived by differentiation of Φ with respect to y. From these considerations, it is concluded that the velocity defined by (15)–(17) represents the flow due to a y-wise free doublet which travels downstream from the origin with velocity U, experiencing viscous diffusion. The solid lines in figure 1 show the distribution of the non-dimensional velocity $\bar{v}_y = v'_y r_0^3/A$ on the axis $\bar{z} = z/r_0$, which is perpendicular to the flow and moves with velocity U. The dashed lines are the distributions on the other perpendicular axis, $\bar{y} = y/r_0$. The parameter k is defined as $k = 4\nu t/r_0^2$, where r_0 is an arbitrary normalizing length.

The flow represented by the velocity components

$$v_x = v'_x\sigma(t), \quad v_y = v'_y\sigma(t), \quad v_z = v'_z\sigma(t) \quad (18)$$

is now considered. In the above equation, $\sigma(t)$ is a step function defined by $\sigma(t) = 0$ when $t < 0$ and $\sigma(t) = 1$ when $t \geqslant 0$. It is related to the impulse function $\delta(t)$ by $d\sigma(t)/dt = \delta(t)$. In view of the property of v'_x that $L_c(v'_x) = 0$, etc., v_x, v_y and v_z are found to give

$$L_c(v_x) = [L_c(v'_x)]\,\sigma(t)+v'_x\delta(t) = v'_x\delta(t) = A\frac{3xy}{R^5}\delta(t) = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$L_c(v_y) = v'_y\delta(t) = A\left(-\frac{1}{R^3}+\frac{3y^2}{R^5}\right)\delta(t) = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$L_c(v_z) = v'_z\delta(t) = A\frac{3yz}{R^5}\delta(t) = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

from which one obtains

$$p = \rho A(y/R^3)\,\delta(t). \quad (19)$$

On putting $A = Y_0/4\pi$ in the above equation, one finds that it is identical with (7). So it is concluded that a y-wise free doublet is shed from the origin at the instant when an impulsive concentrated lift is applied there. The step function used in (18) means that this convecting doublet is born at that instant and kept alive afterwards.

The flow induced by an impulsive line lift on the z axis is obtained from a dense distribution of three-dimensional elementary solutions of this type of equal intensity along this axis. The identity of this flow with the two-dimensional one with concentrated lift at the origin can be confirmed (appendix A).

4. Flow induced by impulsive concentrated drag

On putting

$$v_x' = A\left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{\nu}\frac{\partial f}{\partial t} - \frac{U}{\nu}\frac{\partial f}{\partial x}\right), \quad v_r' = \frac{\partial^2 f}{\partial r\,\partial x}, \tag{20}$$

it is easily verified that these velocities satisfy

$$L_p(v_x') = 0, \quad L_p(v_r') + \nu v_r'/r^2 = 0$$

and the continuity equation

$$\frac{\partial v_x'}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r') = -\frac{A}{\nu}\frac{\partial}{\partial x}[L_p(f)] = 0.$$

The full description of v_x' and v_r' is obtained by substituting (12) into (20):

$$v_x' = A\left\{\left[-\frac{1}{R_1^3} + \frac{3(x-Ut)^2}{R_1^5}\right]\left[1 - \operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right] + \frac{1}{(\nu\pi t)^{\frac{1}{2}}}\exp\left(-\frac{R_1^2}{4\nu t}\right)\left[\frac{1}{R_1^2} - \frac{3(x-Ut)^2}{R_1^4} + \frac{1}{2\nu t}\left(1 - \frac{(x-Ut)^2}{R_1^2}\right)\right]\right\}, \tag{21}$$

$$v_r' = A\left\{\frac{3r(x-Ut)}{R_1^5}\left[1 - \operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right] - \frac{1}{(\nu\pi t)^{\frac{1}{2}}}\exp\left(-\frac{R_1^2}{4\nu t}\right)\frac{r(x-Ut)}{R_1^2}\left(\frac{3}{R_1^3} + \frac{1}{2\nu t}\right)\right\}. \tag{22}$$

In the limit $\nu \to 0$, these reduce to

$$v_x' = A\left\{-\frac{1}{R_1^3} + \frac{3(x-Ut)^2}{R_1^5}\right\}, \quad v_r' = A\frac{3r(x-Ut)}{R_1^5},$$

for which there exists the velocity potential

$$\bar{\phi} = -A(x-Ut)/R_1^3.$$

$\bar{\phi}$ is in turn derived from another potential $\bar{\Phi}$, i.e.

$$\bar{\phi} = \partial\bar{\Phi}/\partial x, \quad \bar{\Phi} = A/R_1.$$

Since the potential $\bar{\Phi}$ is identical with Φ, one finds that the velocities given above represents the flow induced by an x-wise doublet which is shed from the origin and travels downstream with velocity U. Consequently, the velocities given by (21) and (22) correspond to the same doublet experiencing viscous diffusion. The variation of

$\bar{v}_x = v'_x r_0^3/A$ with $\bar{r} = r/r_0$ is identical with the $\bar{v}_y, \bar{z}$ curve in figure 1 while the $\bar{v}_x, \bar{x}$ variation is identical with the $\bar{v}_y, \bar{y}$ curve.

Putting

$$v_x = v'_x \sigma(t), \quad v_r = v'_r \sigma(t), \tag{23}$$

it follows that

$$L_p(v_x) = v'_x \delta(t) = A\left(-\frac{1}{R^3}+\frac{3x^2}{R^5}\right)\delta(t) = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$L_p(v_r) + \nu\frac{v_r}{r^2} = v'_r\delta(t) = A\left(\frac{3xr}{R^5}\right)\delta(t) = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

from which one obtains

$$p = -\rho A(x/R^3)\,\delta(t). \tag{24}$$

This becomes identical with (7) when A is taken as $A = X_0/4\pi$. So, when an impulsive drag is applied at the origin, an x-wise free doublet is shed from the origin and travels downstream with velocity U, experiencing viscous diffusion, and the velocity of the flow is given by (21) and (22). The two-dimensional expression can be derived by distributing densely such axisymmetric elementary solutions on a line perpendicular to the uniform flow in a manner similar to that for concentrated impulsive lift (appendix B).

5. Flow induced by a sinusoidally varying concentrated force

First, sinusoidally varying concentrated drag will be considered in this section. In the preceding section, it is demonstrated that an x-wise free doublet of unit strength is shed from the origin when an impulsive concentrated drag of strength 4π is applied there. When the concentrated drag at the origin varies sinusoidally as $4\pi\exp(i\omega t)$, then at each instant an x-wise free doublet of strength $\exp(i\omega t)$ is shed from the origin and travels downstream with velocity U, experiencing viscous diffusion. On the $+x$ axis, there exists a row of densely distributed x-wise free doublets. It is intended to derive the velocity induced by this row of free doublets. Expressing the velocities as

$$v_x = v''_x \exp(i\omega t), \quad v_r = v''_r \exp(i\omega t),$$

v''_x is calculated first. Using the above-mentioned elementary solution, one obtains the integral expression for v''_x. Putting $R_2^2 = (x-\xi)^2+y^2+z^2$, this becomes after partial integration

$$\begin{aligned} v''_x = \int_0^\infty \Bigg[&\left\{-\frac{1}{R_2^3}+\frac{3(x-\xi)^2}{R_2^5}\right\}\left\{1-\operatorname{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}R_2\right)\right\} \\ &+\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\left\{\frac{1}{R_2^2}-\frac{3(x-\xi)^2}{R_2^4}\right\}\exp\left(-\frac{UR_2^2}{4\nu\xi}\right) \\ &+\left\{1-\frac{(x-\xi)^2}{R_2^2}\right\}\frac{1}{2\pi^{\frac{1}{2}}}\left(\frac{U}{\nu\xi}\right)^{\frac{3}{2}}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\Bigg]\exp\left(-i\omega\frac{\xi}{U}\right)d\xi \\ = &\left[-\frac{\xi-x}{R_2^3}\left\{1-\operatorname{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}R_2\right)\right\}\exp\left(-i\omega\frac{\xi}{U}\right)\right]_0^\infty \\ &-\int_0^\infty\frac{\xi-x}{R_2^3}\,i\,\frac{\omega}{U}\left\{1-\operatorname{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}R_2\right)\right\}\exp\left(-i\omega\frac{\xi}{U}\right)d\xi \end{aligned}$$

$$+\int_0^\infty \frac{\xi-x}{R_2^3}\frac{2}{\pi^{\frac{1}{2}}}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\left(\frac{\xi-x}{R_2}-\frac{R_2}{2\xi}\right)\exp\left(-i\omega\frac{\xi}{U}\right)d\xi$$

$$+\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\left[\left\{\frac{1}{R_2^2}-\frac{\xi(x-\xi)^2}{R_2^4}\right\}+\frac{U}{2\nu\xi}\left\{1-\frac{(x-\xi)^2}{R_2^2}\right\}\right]\exp\left(-i\omega\frac{\xi}{U}\right)\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)d\xi$$

$$=-\frac{x}{R^3}+\int_0^\infty -i\frac{\omega}{U}\left[\frac{\xi-x}{R_2^3}\left\{1-\operatorname{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}R_2\right)\right\}\right.$$

$$\left.+\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{x-\xi}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\omega\frac{\xi}{U}\right)d\xi$$

$$+\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\left[-\frac{2(\xi-x)^2}{R_2^4}-\frac{1}{2\xi}\frac{\xi-x}{R_2^2}+\frac{1}{R_2^2}\right.$$

$$\left.+\frac{U}{2\nu\xi}\left\{1-\frac{(\xi-x)^2}{R_2^2}\right\}\right]\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\omega\frac{\xi}{U}\right)d\xi$$

$$+\int_0^\infty\left(\frac{U}{(\nu\pi\xi)}\right)^{\frac{1}{2}}\left(-i\frac{\omega}{U}\right)\frac{(\xi-x)}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\omega\frac{\xi}{U}\right)d\xi.$$

Partial integration of the last term of this equation gives

$$\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\left(-i\frac{\omega}{U}\right)\frac{(\xi-x)}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\omega\frac{\xi}{U}\right)d\xi$$

$$=\left[\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{\xi-x}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\omega\frac{\xi}{U}\right)\right]_0^\infty$$

$$-\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\left[\frac{1}{R_2^2}-\frac{2(\xi-x)^2}{R_2^4}-\frac{U}{4\nu}\frac{\xi-x}{R_2^2}\left\{\frac{2(\xi-x)}{\xi}-\frac{R_2^2}{\xi^2}\right\}-\frac{1}{2R_2^2}\frac{\xi-x}{\xi}\right]$$

$$\times\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\frac{\omega\xi}{U}\right)d\xi.$$

The first term vanishes because

$$\lim_{\xi\to 0}\{\xi^{-\frac{1}{2}}\exp(-UR_2^2/4\nu\xi)\}=0,$$

whence v_x'' becomes

$$v_x''=-\frac{x}{R^3}+\int_0^\infty\left(-i\frac{\omega}{U}\right)\left[\frac{\xi-x}{R_2^3}\left\{1-\operatorname{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}R_2\right)\right\}\right.$$

$$\left.+\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{x-\xi}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\frac{\omega}{U}\xi\right)d\xi$$

$$+\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{U}{4\nu}\left(\frac{1}{\xi}+\frac{x}{\xi^2}\right)\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\frac{\omega}{U}\xi\right)d\xi.$$

The last term of the above equation reduces to

$$I=\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{U}{4\nu}\left(\frac{1}{\xi}+\frac{x}{\xi^2}\right)\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\exp\left(-i\frac{\omega}{U}\xi\right)d\xi$$

$$=\exp(kx)\int_0^\infty\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{U}{4\nu}\left(\frac{1}{\xi}+\frac{x}{\xi^2}\right)\exp\left(-\frac{\beta^2}{2k}\xi-\frac{k}{2}\frac{R^2}{\xi}\right)d\xi,$$

in which k has been replaced by $U/2\nu$, β^2 by $k^2+2ik\omega/U$ and $R^2 = x^2+r^2$. Furthermore, on putting $\xi = (kR/\beta)\eta$, I becomes

$$I = \exp(kx)\int_0^\infty \left(\frac{U}{\nu\pi}\right)^{\frac{1}{2}} \frac{U}{4\nu}\left(\frac{\beta}{k}\frac{1}{R\eta}\right)^{\frac{1}{2}}\left(\frac{1}{\eta}+\frac{\beta x}{kR\eta^2}\right)\exp\left\{-\frac{\beta}{2}R\left(\eta+\frac{1}{\eta}\right)\right\}d\eta.$$

Using the relations

$$K_{\frac{1}{2}}(\beta R) = \left(\frac{\pi}{2\beta R}\right)^{\frac{1}{2}}\exp(-\beta R) = \frac{1}{2}\int_0^\infty \eta^{-\frac{1}{2}}\exp\left\{-\frac{\beta R}{2}\left(\eta+\frac{1}{\eta}\right)\right\}d\eta,$$

$$K_{\frac{3}{2}}(\beta R) = \frac{\beta R+1}{\beta R}\left(\frac{\pi}{2\beta R}\right)^{\frac{1}{2}}\exp(-\beta R) = \frac{1}{2}\int_0^\infty \eta^{-\frac{5}{2}}\exp\left\{-\frac{\beta R}{2}\left(\eta+\frac{1}{\eta}\right)\right\}d\eta,$$

one obtains

$$I = \exp(kx)\left(\frac{U}{\nu\pi}\right)^{\frac{1}{2}}\frac{U}{4\nu}\left(\frac{\beta}{kR}\right)^{\frac{1}{2}}\left\{2K_{\frac{1}{2}}(\beta R)+\frac{2\beta x}{kR}K_{\frac{3}{2}}(\beta R)\right\}$$

$$= \left\{\frac{x}{R^3}+\frac{\beta}{R^2}\left(\frac{k}{\beta}R+x\right)\right\}\exp(kx-\beta R).$$

Finally, v_x'' reduces to

$$v_x'' = -\frac{x}{R^3}+\int_0^\infty -i\frac{\omega}{U}\left[\frac{\xi-x}{R_2^5}\left\{1-\text{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\}\right.$$
$$\left. -\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{x-\xi}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\frac{\omega}{U}\xi\right)d\xi$$
$$+\left\{\frac{x}{R^3}+\frac{\beta}{R^2}\left(\frac{k}{\beta}R+x\right)\right\}\exp(kx-\beta R). \tag{25}$$

In an entirely similar manner, one obtains

$$v_r'' = -\frac{r}{R^3}+\int_0^\infty i\frac{\omega}{U}\left[\frac{r}{R_2^3}\left\{1-\text{erfc}\left(\frac{1}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\}-\left(\frac{U}{\nu\pi\xi}\right)^{\frac{1}{2}}\frac{r}{R_2^2}\right]$$
$$\times\exp\left(-i\omega\frac{\xi}{U}\right)d\xi+\left(\frac{r}{R^3}+\beta\frac{r}{R^2}\right)\exp(kx-\beta R). \tag{26}$$

The first terms of (25) and (26) represent the potential flow due to a source at the origin, whose potential is given by $\phi = 1/R$. The second terms are interpreted as follows. The velocities derived from f as defined by (12), i.e.

$$v_x = \frac{\partial f}{\partial x} = -\frac{x}{R_1^3}\left\{1-\text{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{1}{(\pi\nu t)^{\frac{1}{2}}}\frac{x}{R_1^2}\exp\left(-\frac{R_1^2}{4\nu t}\right), \tag{27}$$

$$v_r = \frac{\partial f}{\partial r} = -\frac{r}{R_1^3}\left\{1-\text{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{1}{(\pi\nu t)^{\frac{1}{2}}}\frac{r}{R_1^2}\exp\left(-\frac{R_1^2}{4\nu t}\right), \tag{28}$$

satisfy

$$L_p(v_x) = 0,\quad L_p(v_r)+\nu v_r/r^2 = 0.$$

So the flow represented by these velocity components does not contribute to the static pressure. In the limit $\nu\to 0$, these velocities become

$$v_x = -x/R_1^3,\quad v_r = -r/R_1^3$$

and have a velocity potential $\phi = 1/R_1$ representing a moving source with velocity U. Hence the second terms in (25) and (26) correspond to the velocity induced by the densely distributed row of sources on the $+x$ axis, which is composed of the moving sources consecutively leaving the origin and experiencing viscous diffusion.

The third terms of (25) and (26) correspond to the Oseenlet in a narrow sense.

The velocities given by (27) and (28) do not satisfy the continuity equation, nor does the Oseenlet in a narrow sense. However, in (25) and (26), the residuals cancel each other and together satisfy the continuity equation. This aspect is similar in the two-dimensional flow.

In the limit $\omega \to 0$, $\beta = U/2\nu$ and (25) and (26) reduce to

$$v'_x = -\frac{x}{R^3} + \left\{\frac{x}{R^3} + \frac{U}{2\nu}\frac{R+x}{R^2}\right\}\exp\left\{\frac{U}{2}(x-R)\right\},$$

$$v'_r = -\frac{r}{R^3} + \left\{\frac{r}{R^3} + \frac{U}{2\nu}\frac{r}{R^2}\right\}\exp\left\{\frac{U}{2}(x-R)\right\}.$$

These are the well-known elementary solutions for steady flow (Moore 1964).

In addition to the drag solution considered above, the elementary solution for the case of sinusoidally varying concentrated lift must be found. When a concentrated lift which varies as $4\pi \exp(i\omega t)$ is applied at the origin, y-wise free doublets of strength $\exp(i\omega t)$ are shed consecutively from the origin and form a densely distributed row of y-wise free doublets whose intensity decreases with increasing distance from the origin owing to viscous diffusion. Writing the velocities induced by this row of free doublets as

$$v_x, v_y, v_z = (v''_x, v''_y, v''_z)\exp(i\omega t), \tag{29}$$

v''_x, v''_y and v''_z are easily found to have the following form by an extension of the approach in § 3:

$$v''_x = \int_0^\infty \left[\frac{3y(x-\xi)}{R_2^5}\left\{1-\operatorname{erfc}\left(\frac{R_2}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\} - \left(\frac{U}{\pi\nu\xi}\right)^{\frac{1}{2}}\frac{3y(x-\xi)}{R_2^4}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right.$$
$$\left. -\frac{1}{2\pi^{\frac{1}{2}}}\left(\frac{U}{\nu\xi}\right)^{\frac{3}{2}}\frac{y(x-\xi)}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\frac{\omega\xi}{U}\right)d\xi, \tag{30}$$

$$v''_z = \int_0^\infty \left[\frac{3yz}{R_2^5}\left\{1-\operatorname{erfc}\left(\frac{R_2}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\} - \left(\frac{U}{\pi\nu\xi}\right)^{\frac{1}{2}}\frac{3yz}{R_2^4}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right.$$
$$\left. -\frac{1}{2\pi^{\frac{1}{2}}}\left(\frac{U}{\nu\xi}\right)^{\frac{3}{2}}\frac{yz}{R_2^2}\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\frac{\omega\xi}{U}\right)d\xi, \tag{31}$$

$$v''_y = \int_0^\infty \left[\left(-\frac{1}{R_2^3}+\frac{3y^2}{R_2^5}\right)\left\{1-\operatorname{erfc}\left(\frac{R_2}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\} + \left(\frac{U}{\pi\nu\xi}\right)^{\frac{1}{2}}\left(\frac{1}{R_2^2}-\frac{3y^2}{R_2^4}\right)\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right.$$
$$\left. +\frac{1}{2\pi^{\frac{1}{2}}}\left(\frac{U}{\nu\xi}\right)^{\frac{3}{2}}\left(1-\frac{y^2}{R_2^2}\right)\exp\left(-\frac{U}{4\nu\xi}R_2^2\right)\right]\exp\left(-i\frac{\omega\xi}{U}\right)d\xi. \tag{32}$$

Conversion of these expressions into ones like those for v_x'' and v_r'' in (25) and (26) seems to be highly laborious work, except for the x-wise component. The same sequence of manipulations as was done for v_x'' to derive (25) yields the following result for the v_x'' given by (30):

$$v_x'' = -\frac{y}{R^3}+\int_0^\infty i\frac{\omega}{U}\left[\frac{y}{R_2^3}\left\{1-\operatorname{erfc}\left(\frac{R_2}{2}\left(\frac{U}{\nu\xi}\right)^{\frac{1}{2}}\right)\right\}-\left(\frac{U}{\pi\nu\xi}\right)^{\frac{1}{2}}\frac{y}{R_2^2}\exp\left(-\frac{UR_2^2}{4\nu\xi}\right)\right]$$
$$\times\exp\left(-i\frac{\omega\xi}{U}\right)d\xi+\left(\frac{y}{R^3}+\frac{\beta y}{R^2}\right)\exp\left(kx-\beta R\right). \quad (33)$$

The first term is due to the potential vortex, the second term to the densely distributed row of diffusing free vortices on the $+x$ axis and the last term to an Oseenlet in a narrow sense.

For the purpose of evaluating lift on the basis of Oseen's approximation, it is possible to formulate a theory such as lifting-surface theory using (30)–(32).

6. Conclusion

The three-dimensional Oseen flow which occurs when a concentrated force is applied at a point in an otherwise uniform stream has been analysed. The velocity field produced by a concentrated impulsive force has been derived and it has been demonstrated that the flow field may be represented by a diffusing free doublet travelling with the velocity of the uniform stream.

Also, the flow due to a sinusoidally varying concentrated force has been obtained as a superposition of solutions for an impulsive force and converted into a form which yields a clear physical picture of the structure of the flow.

Appendix A. Derivation of two-dimensional flow from the three-dimensional elementary solution for concentrated lift

It is intended to show that a dense distribution of three-dimensional elementary solutions of unit strength ($Y_0 = 1$ and consequently $A = 1/4\pi$) on the z axis ($\infty > z > -\infty$) yields the two-dimensional flow when a concentrated impulsive lift is applied at the origin.

The x-wise velocity v_x' of the flow is

$$v_x' = \frac{1}{4\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{\partial}{\partial y}\left[\frac{1}{R_1}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]dz$$
$$= \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^\infty\left[-\frac{y}{R_1^3}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\frac{y}{R_1^2}\exp\left(-\frac{R_1^2}{4\nu t}\right)\right]dz = \frac{y}{2\pi}\frac{\partial I_1}{\partial x}.$$

Partial integration using the relation

$$\int\frac{1}{R_1^3}dz = \frac{z}{r_1^2}\frac{1}{(r_1^2+z^2)^{\frac{1}{2}}},\quad r_1^2 = (x-Ut)^2+y^2,$$

yields

$$I_1 = \left[-\frac{z}{r_1^2} \frac{1}{(r_1^2+z^2)^{\frac{1}{2}}} \left\{1-\operatorname{erfc}\left(\frac{(r_1^2+z_1^2)^{\frac{1}{2}}}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]_0^\infty$$

$$+\frac{2}{\pi^{\frac{1}{2}}} \frac{1}{2(\nu t)^{\frac{1}{2}}} \int_0^\infty \frac{1}{r_1^2+z^2}\left(1+\frac{z^2}{r_1^2}\right) \exp\left(-\frac{r_1^2+z^2}{4\nu t}\right) dz$$

$$= -\frac{1}{r_1^2}+\frac{1}{r_1^2}\exp\left(-\frac{r_1^2}{4\nu t}\right)\frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty \exp\left(-\frac{z^2}{4\nu t}\right)\frac{dz}{2(\nu t)^{\frac{1}{2}}}$$

$$= -\frac{1}{r_1^2}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\left[1-\operatorname{erf}\eta\right]_0^\infty\right\} = -\frac{1}{r_1^2}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}.$$

Then one finds

$$v_x' = \frac{1}{2\pi}\frac{\partial}{\partial x}\left[-\frac{y}{r_1^2}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}\right].$$

This velocity is exactly the x component of the two-dimensional flow produced when a two-dimensional concentrated impulsive lift is applied at the origin.

Next the y component will be examined. The y-wise velocity produced when a concentrated impulsive lift is applied at the origin is given by (14), and as f satisfies $L_p(f) = 0$, v_y' is given by

$$v_y' = -\frac{1}{4\pi}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial z^2}\right).$$

Integration of this velocity yields

$$v_y' = -\frac{1}{2\pi}\left(\int_0^\infty \frac{\partial^2 f}{\partial x^2}dz+\left|\frac{\partial f}{\partial z}\right|_0^\infty\right).$$

The relation

$$\frac{\partial f}{\partial z} = -\frac{z}{R_1^3}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\frac{z}{R_1^2}\exp\left(-\frac{R_1^2}{4\nu t}\right)$$

then gives

$$[\partial f/\partial z]_0^\infty = 0,$$

and as a consequence

$$v_y' = \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^\infty -\frac{\partial f}{\partial x}dz = \frac{1}{2\pi}\frac{\partial I_2}{\partial x}.$$

Furthermore, since

$$\frac{\partial f}{\partial x} = -\frac{(x-Ut)}{R_1^3}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\frac{(x-Ut)}{R_1^2}\exp\left(-\frac{R_1^2}{4\nu t}\right),$$

$$I_2 = (x-Ut)\left[\left[\frac{z}{r_1^2}\frac{1}{(r_1^2+z^2)^{\frac{1}{2}}}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]_0^\infty\right.$$

$$\left.-\frac{1}{r_1^2}\exp\left(\frac{-r_1^2}{4\nu t}\right)\frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty \exp\left(-\frac{z^2}{4\nu t}\right)\frac{dz}{2(\nu t)^{\frac{1}{2}}}\right] = \frac{x-Ut}{r_1^2}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\},$$

and v_y' finally becomes

$$v_y' = \frac{1}{2\pi}\frac{\partial}{\partial x}\left[\frac{x-Ut}{r_1^2}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}\right].$$

This is exactly the two-dimensional y-wise velocity produced when a two-dimensional concentrated impulsive lift is applied at the origin.

The z-wise velocity of the flow under consideration is easily found to be zero; since the z component of the three-dimensional elementary solution for the present case is an odd function of z, the integral with respect to z is automatically zero.

Appendix B. Derivation of two-dimensional flow from the three-dimensional elementary solution for concentrated drag

It is intended to show that a dense distribution of three-dimensional elementary solutions on the z axis yields the two-dimensional flow when a concentrated impulsive drag is applied at the origin.

The x-wise velocity v_x' of the flow is

$$v_x' = \frac{1}{2\pi}\left[\int_0^\infty \left\{-\frac{1}{R_1^3}+\frac{3(x-Ut)^2}{R_1^5}\right\}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}dz\right.$$

$$\left. +\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\int_0^\infty \exp\left(-\frac{R_1^2}{4\nu t}\right)\left[\left\{\frac{1}{R_1^2}-\frac{3(x-Ut)^2}{R_1^4}\right\}+\left\{1-\frac{(x-Ut)^2}{R_1^2}\right\}\frac{1}{2\nu t}\right]dz\right]$$

$$= \frac{1}{2\pi}\left\{\left[\left\{-\frac{z}{r_1^2R_1}+\frac{(x-Ut)^2}{r_1^2}\left(\frac{2z}{r_1^2R_1}+\frac{z}{R_1^3}\right)\right\}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]_0^\infty\right.$$

$$+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\exp\left(-\frac{r_1^2}{4\nu t}\right)\int_0^\infty\left[\frac{z^2}{r_1^2R_1^2}-\frac{(x-Ut)^2}{r_1^2}\left(\frac{2z^2}{r_1^2R_1^2}+\frac{z^2}{R_1^4}\right)\right.$$

$$\left.\left.+\frac{1}{R_1^2}-\frac{3(x-Ut)^2}{R_1^4}+\left\{1-\frac{(x-Ut)^2}{R_1^2}\right\}\frac{1}{2\nu t}\right]\exp\left(-\frac{z^2}{4\nu t}\right)dz\right\}$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{r_1^2}+\frac{2(x-Ut)^2}{r_1^4}\right\}\right.$$

$$\left.+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\exp\left(-\frac{r_1^2}{4\nu t}\right)\int_0^\infty\left\{\frac{1}{r_1^2}-2\frac{(x-Ut)^2}{r_1^4}\right\}\exp\left(-\frac{z^2}{4\nu t}\right)dz\right]$$

$$+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\exp\left(-\frac{r_1^2}{4\nu t}\right)\int_0^\infty\left[\left\{\frac{(x-Ut)^2}{R_1^2r_1^2}-\frac{2(x-Ut)^2}{R_1^4}\right\}\right.$$

$$\left.+\left\{1-\frac{(x-Ut)^2}{R_1}\right\}\frac{1}{2\nu t}\exp\left(-\frac{z^2}{4\nu t}\right)dz\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{r_1^2}+\frac{2(x-Ut)^2}{r_1^4}\right\}\right.$$

$$+\left\{\frac{1}{r_1^2}-2\frac{(x-Ut)^2}{r_1^4}\right\}\exp\left(-\frac{r_1^2}{4\nu t}\right)\left[1-\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}\right)\right]_0^\infty$$

$$+\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\exp\left(-\frac{r_1^2}{4\nu t}\right)\left[\left[-\frac{(x-Ut)^2}{r_1^2}\frac{z^2}{R_1^2}\exp\left(-\frac{z^2}{4\nu t}\right)\right]_0^\infty\right.$$

$$\left.\left.+\int_0^\infty\left\{-\frac{(x-Ut)^2}{r_1^2}\frac{z}{R_1^2}\frac{2z}{4\nu t}+\frac{1}{2\nu t}\left(1-\frac{(x-Ut)^2}{R_1^2}\right)\right\}\exp\left(-\frac{z^2}{4\nu t}\right)dz\right]\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{r_1^2}+\frac{2(x-Ut)^2}{r_1^4}\right\}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}\right.$$
$$\left.+\frac{1}{2\nu t}\exp\left(-\frac{r_1^2}{4\nu t}\right)\frac{y^2}{r_1^2}\left[1-\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}\right)\right]_0^\infty\right]$$
$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{r_1^2}+\frac{2(x-Ut)^2}{r_1^4}\right\}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}+\frac{1}{2\nu t}\frac{y^2}{r_1^2}\exp\left(-\frac{r_1^2}{4\nu t}\right)\right].$$

This is exactly the two-dimensional x-wise velocity produced when a two-dimensional concentrated impulsive drag is applied at the origin.

The y-wise velocity is derived similarly:

$$v'_y = \frac{1}{2\pi}\left[\int_0^\infty \frac{3y(x-Ut)}{R_1^5}\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}dz\right.$$
$$\left.-\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}\int_0^\infty\left\{\frac{3y(x-Ut)}{R_1^4}+\frac{1}{2\nu t}\frac{y(x-Ut)}{R_1^2}\right\}\exp\left(-\frac{R_1^2}{4\nu t}\right)dz\right]$$
$$= \frac{1}{2\pi}\left[\left[y(x-Ut)\left(\frac{2z}{r_1+R_1}+\frac{z}{r_1^2R_1^3}\right)\left\{1-\operatorname{erfc}\left(\frac{R_1}{2(\nu t)^{\frac{1}{2}}}\right)\right\}\right]_0^\infty\right.$$
$$\left.-\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}y(x-Ut)\int_0^\infty\left(\frac{2z^2}{r_1^4R_1^2}+\frac{z^2}{r_1^2R_1^4}+\frac{3}{R_1^4}+\frac{1}{2\nu t}\frac{1}{R_1^2}\right)\exp\left(-\frac{R_1^2}{4\nu t}\right)dz\right]$$
$$= \frac{1}{2\pi}\left[\frac{2y(x-Ut)}{r_1^4}-\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}y(x-Ut)\exp\left(-\frac{r_1^2}{4\nu t}\right)\int_0^\infty\right.$$
$$\left.\times\left(\frac{2}{r_1^4}+\frac{2}{R_1^4}-\frac{1}{r_1^2R_1^2}+\frac{1}{2\nu t}\frac{1}{R_1^2}\right)\exp\left(-\frac{z^2}{4\nu t}\right)dz\right]$$
$$= \frac{1}{2\pi}\left[\frac{2y(x-Ut)}{r_1^4}-\frac{2}{\pi^{\frac{1}{2}}}\frac{1}{2(\nu t)^{\frac{1}{2}}}y(x-Ut)\exp\left(-\frac{r_1^2}{4\nu t}\right)\left\{\left[\frac{1}{r_1^2}\frac{z}{R_1^2}\exp\left(-\frac{z^2}{4\nu t}\right)\right]_0^\infty\right.\right.$$
$$\left.\left.+\int_0^\infty\left(\frac{1}{2\nu t}+\frac{2}{r_1^4}\right)\exp\left(-\frac{z^2}{4\nu t}\right)dz\right\}\right]$$
$$= \frac{1}{2\pi}\left[\frac{2y(x-Ut)}{r_1^4}-y(x-Ut)\exp\left(-\frac{r_1^2}{4\nu t}\right)\left(\frac{1}{2\nu t}+\frac{2}{r_1^4}\right)\left[1-\operatorname{erfc}\left(\frac{z}{2(\nu t)^{\frac{1}{2}}}\right)\right]_0^\infty\right]$$
$$= \frac{1}{2\pi}\left[\frac{2y(x-Ut)}{r_1^4}\left\{1-\exp\left(-\frac{r_1^2}{4\nu t}\right)\right\}-\frac{1}{2\nu t}y(x-Ut)\exp\left(-\frac{r_1^2}{4\nu t}\right)\right].$$

The z-wise velocity of the flow under consideration is found to be zero, since the z component of the three-dimensional elementary solution for the present case is zero.

REFERENCES

MOORE, F. M. 1964 *Theory of Laminar Flows*, p. 96. Princeton University Press.

MURATA, S., MIYAKE, Y. & TSUJIMOTO, Y. 1977 *Bull. Japan Soc. Mech. Engrs* **20**, 819.

SHEN, S. F. & CRIMI, P. J. 1965 *J. Fluid Mech.* **23**, 585.

TSUJIMOTO, Y., MURATA, S., MIYAKE, Y. & YAMAMOTO, F. 1978 *Trans. Japan Soc. Mech. Engrs* (in Japanese) (to be published).